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Scaling law and asymptotic distribution of Lyapunov exponents in conservative dynamical systems with many degrees of freedom

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Abstract. We study by numerical means the infinite product of $2N\times 2N$ conservative random matrices which mimics the chaotic behaviour of Hamiltonian systems with $N+1$ degrees of freedom made of weakly nearest-neighbour coupled oscillators. The maximum Lyapunov exponent λ_1 exhibits a power-law behaviour as a function of the coupling constant ε: $\lambda_1 \sim \varepsilon^{\beta}$ with either $\beta=\frac{1}{2}$ or $\beta=\frac{2}{3}$, depending on the probability distribution of the matrix elements. These power laws do not depend on N and moreover increasing N, λ_1 rapidly tends to an asymptotic value λ_1^* which only depends on ε and on the kind of probability distribution chosen for building up the matrices. We also compute the spectrum of the Lyapunov exponents and show that it has a thermodynamic limit of large N.

This suggests the existence of a Kolmogorov entropy per degree of freedom proportional to λ_1^*.

1. Introduction

The chaotic behaviour of dynamical systems can be investigated by means of the properties of the maximal characteristic Lyapunov exponents λ_1, which are related roughly speaking to the divergence of nearby orbits.

In this paper, we are interested in conservative dynamical systems, which are smoothly dependent on a parameter ε and integrable for vanishing ε, with a large number N of degrees of freedom.

Let us recall that Benettin (1984) has found for some two-dimensional systems (the billiards)

$$\lambda_1 \sim \varepsilon^{\beta} \tag{1}$$

with $\beta=\frac{1}{2}$.

Moreover Rechester *et al* (1979) in a simple conservative map on the two-dimensional torus obtained the scaling law (1) with $\beta=\frac{2}{3}$.

These power laws might indicate universal features in conservative systems as were also found for 2×2 random matrices whose determinants have a value $+1$. The powers however are either $\beta=\frac{1}{2}$ or $\beta=\frac{2}{3}$ depending on the class of probability laws used in building up the matrices (Benettin 1984).

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In this paper we are interested in the infinite product of $2N\times 2N$ conservative random matrices built up in such a way that one has a rough but not trivial approximation of a Hamiltonian system made of $(N+1)$ weakly nearest-neighbour coupled oscillators.

We look at the ε dependence of the maximal Lyapunov exponent (say λ_1) for systems with an arbitrary N and thus show that power law (1) with $\beta=\frac{1}{2}$ or $\beta=\frac{2}{3}$ obtained in the case $N=1$ (Benettin 1984) still holds for any value of N.

It is then interesting to discover if a 'thermodynamic' limit does exist for large N. This might involve a great simplification in investigating properties of large systems.

We shall see, for example, that the Lyapunov exponents $\lambda_i (\lambda_i \geq \lambda_{i+1}$ with $i=1,\ldots,2N)$ only depend on i/N for N large enough:

$$\begin{aligned} &\lambda_i \simeq \lambda_1^* f(i/N) \\ &\lambda_1^* = \lim_{N\to\infty} \lambda_1(N). \end{aligned} \tag{2}$$

One can thus deduce that there is a Kolmogorov entropy density.

The Kolmogorov entropy for Hamiltonian systems under the hypothesis that the λ_i do not depend on initial conditions is (Pesin 1976)

$$H_N = \sum_{i=1}^{2N} \lambda_i \theta(\lambda_i) \tag{3}$$

and becomes in the thermodynamical limit by means of (2)

$$H_N \simeq \sum_{i=1}^{2N} \lambda_1^* f(i/N)\theta(f(i/N)) \simeq hN\lambda_1^* \tag{4}$$

where h depends on the particular form of the function $f(i/N)$.

Behaviour (2) was also found in some dissipative (Manneville 1983, Pomeau *et al* 1984) and conservative (Livi *et al* 1986) systems.

We shall show that λ_1 fast approaches its asymptotic value λ_1^*, and that λ_i/λ_1 is linear in our case.

In § 2 we explain the details of the construction of our random matrices; in § 3 we discuss the scaling laws of λ_1 and in § 4 we present the results concerning the existence of a thermodynamic limit.

2. Random matrices products

The applicability of (1) to dynamical systems with $N\geq 2$ is not quite trivial because of the dependence on initial conditions. Different behaviours are possible according to the starting region in the phase space (Contopoulos *et al* 1978, Pettini and Vulpiani 1984) almost for negligible Arnold diffusion (Arnold and Avez 1968), namely for times which may become very large compared to those observed.

We wish moreover to deal with typical models which may be representative of a large class of perturbation forms.

Let us therefore consider a sequence of $2N\times 2N$ random matrices $A_\varepsilon(k)$:

$$A_\varepsilon(k) = \left(\begin{array}{c|c} \mathbb{1} & \mathbb{1} \\ \hline \varepsilon a(k) & \mathbb{1}+\varepsilon a(k) \end{array}\right). \tag{5}$$

Here $\mathbb{1}$ is the $N\times N$ identity matrix and a is a symmetric $N\times N$ random matrix.

A_ε can be regarded as the linearised evolution matrix of the $2N$-dimensional map:

$$\begin{aligned} \boldsymbol{q}(n+1) &= \boldsymbol{q}(n)+\boldsymbol{p}(n) \\ \boldsymbol{p}(n+1) &= \boldsymbol{p}(n)+\varepsilon\nabla F[\boldsymbol{q}(n+1)] \end{aligned} \tag{6}$$

where $\nabla = (\partial/\partial q_1, \ldots, \partial/\partial q_N)$. Note that (6) is volume-preserving and $\det A_\varepsilon = 1$ for construction.

The problem of the product of random matrices

$$\prod_{k=1}^{M} A_\varepsilon(k) \tag{7}$$

is indeed related in a natural way to dynamical systems and (6) can be seen as the Poincaré map of a Hamiltonian system with $(N+1)$ degrees of freedom (Lichtenberg and Lieberman 1983).

Moreover it is easy to see (Chirikov 1982) that the Kolmogorov entropy H of the mapping (6) and the Kolmogorov entropy $\tilde{H}$ of the original Hamiltonian system are related by

$$H = T\tilde{H}$$

where $T = \langle(t_{n+1} - t_n)\rangle$ if the trajectory crosses, at $t = t_n$ for the nth time, the surface of the section. Thus one sees that the qualitative and quantitative 'degree of chaoticity' does not change with the reduction of the original Hamiltonian system to the mapping (6).

Let us also note that (6) are the equations of N uncoupled oscillators when ε vanishes.

The randomness of $A_\varepsilon(k)$ mimics the chaoticity of the trajectory generated by (6) in the phase space, which is in principle deterministic. Thus we think that (7) is the first crude (but not trivial) approximation for describing the dynamics of coupled oscillators.

In order to represent nearest-neighbour couplings, we assume for the symmetric matrix a the form

$$a = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \ldots & 0 & a_{1,N} \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \ldots & 0 \\ 0 & & & & & \vdots \\ \vdots & & & & & 0 \\ 0 & & & & & a_{N-1,N} \\ a_{N,1} & 0 \ldots & \ldots & 0 & a_{N,N-1} & a_{N,N} \end{pmatrix}. \tag{8}$$

The non-zero elements of a are random variables distributed according to different probability laws

$$a_{ij} = \tfrac{1}{2}(x_m)^\alpha + \bar{x}. \tag{9}$$

Here α is an odd integer and $\bar{x}$ the average of all non-zero elements a_{ij}.

We shall consider the following cases labelled by m; $m=1$: x_1 is uniformly distributed in the interval $(-1, 1)$, $m=2$: x_2 is Gaussian with zero mean value and variance $\sigma^2 = 1$.

Let us finally remark that it is realistic to assume that the interaction Hamiltonian $F(\boldsymbol{q})$ among the oscillators should be of the form

$$F(\boldsymbol{q}) = \sum_{j=1}^{N} \tilde{F}(q_{j+1} - q_j) \tag{10}$$

where the sum runs over the N oscillators labelled by the index j.

This assumption implies a further constraint on the elements of the $N \times N$ matrix a

$$\sum_{j=1}^{N} a_{ij} = 0 \tag{11}$$

which is easy to check by a simple calculation.

In the following we shall consider both cases where either constraint (11) is imposed or it is not.

3. Maximal Lyapunov exponent

The maximal Lyapunov exponent λ_1 is defined for (7) as (Benettin *et al* 1980a, b):

$$\begin{aligned} \lambda_1 &= \lim_{M\to\infty} M^{-1} \ln |\boldsymbol{\zeta}(M)| \\ \boldsymbol{\zeta}(M) &= \left(\prod_{k=1}^{M} A_\varepsilon(k) \right) \boldsymbol{\zeta}(0) \end{aligned} \tag{12}$$

where $\boldsymbol{\zeta}(0)$ is a generical vector of $\mathbb{R}^{2N}$.

We have numerically found that the $N=1$ results hold without any observable dependence on N and on the particular probability distribution chosen as shown in figure 1.

Thus one has

$$\lambda_1 \simeq C(N)\varepsilon^{\beta} \tag{13}$$

with $\beta = \frac{1}{2}$ if $\bar{x} \neq 0$, $\beta = \frac{2}{3}$ if $\bar{x} = 0$.

λ_1 has been numerically computed, as usual, by following naively the definition (12), with a random choice of $\boldsymbol{\zeta}(0)$. The maximum number of iterations M used is 5×10^4 and for small values of ε, 2×10^5.

In any case the quantity

$$(1/n) \ln \left| \left(\prod_{k=1}^{n} A_\varepsilon(k) \right) \boldsymbol{\zeta}(0) \right|$$

was found to exhibit fluctuations of at most 1-2% in the range $M/10 < n < M$. As a random number generator the internal generator of a VAX-11/780 was used. In the case with the constraint (11) we have generated the off-diagonal matrix element a_{ij} and then the a_{ii} have been determined by the constraint.

The maximal Lyapunov exponent is known to tend quickly to a constant as N increases for chains of oscillators with Lennard-Jones-type interactions at a fixed energy constant (Casartelli *et al* 1976).

It is thus not surprising that the prefactor $C(N)$ tends to an asymptotic value $C(\infty)$ which is reached in practice for not too large N (i.e. $N \gtrsim 10$).

Figure 2 shows that the behaviour is quite well fitted by $C(N) = C(\infty) - b/N$ where the constant b is very small if $\bar{x} \neq 0$ and constraint (11) is not satisfied.

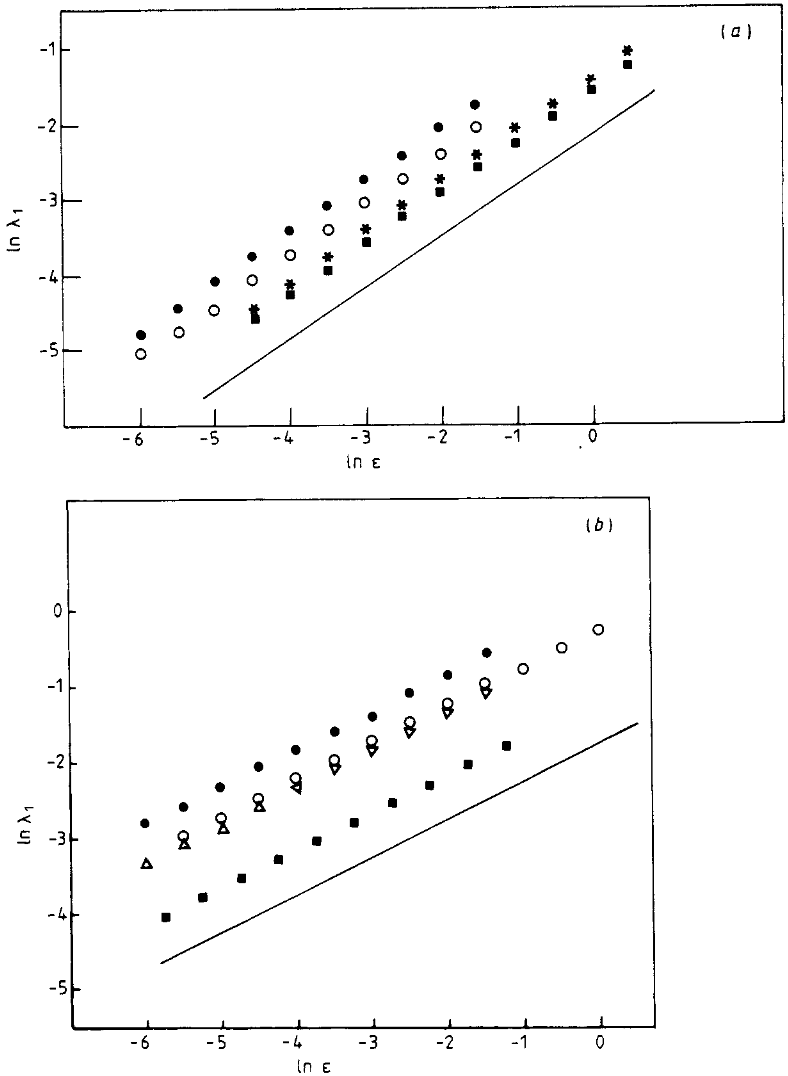

Figure 1. ln λ_1 plotted against ln ε.: (*a*) $\bar{x}=0$; ○: $N=4$, $\alpha=1$, $m=1$, constraint (11) imposed; ●: $N=6$, $\alpha=1$, $m=2$, constraint (11) not imposed; $*$: $N=4$, $\alpha=3$, $m=1$, constraint (11) not imposed; ■: $N=4$, $\alpha=5$, $m=1$, constraint (11) not imposed. The line indicates the slope $\frac{2}{3}$; (*b*) $\bar{x}\neq 0$; ●: $N=5$, $\alpha=3$, $m=2$, $\bar{x}=0.5$, constraint (11) not imposed; Δ: $N=7$, $\alpha=3$, $m=2$, $\bar{z}=0.5$, constraint (11) imposed; ○: $N=4$, $\alpha=3$, $m=1$, $\bar{x}=0.2$, constraint (11) not imposed; ■: $N=8$, $\alpha=1$, $m=1$, $\bar{x}=0.1$, constraint (11) imposed. The line indicates the slope $\frac{1}{2}$.

4. Distribution of Lyapunov exponents in the large N limit

We have shown in § 3 that λ_1 quickly approaches an asymptotic value with increasing N.

This result may indicate the existence of a sort of 'thermodynamic limit' when $N\to\infty$ for many properties of the dynamics generated by (6).

It is then useful to compute the set of all the Lyapunov exponents $\{\lambda_i\}$ which give a good (even if not complete) description of a dynamical system.

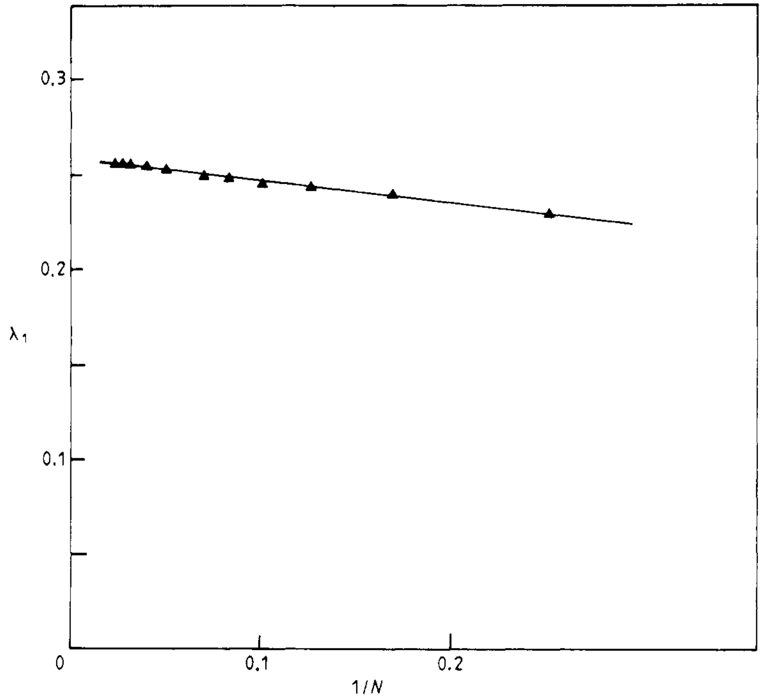

Figure 2. λ_1 plotted against $1/N$ without imposing constraint (11). $\alpha=3$, $\bar{x}=0$, $\varepsilon=0.1$ and $m=2$.

Let us recall the definition of the $\{\lambda_i\}$ where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{2N}$ for a dynamical system $\boldsymbol{x}(n+1)=\boldsymbol{g}(\boldsymbol{x}(n))$; $\boldsymbol{g}$, $\boldsymbol{x} \in \mathbb{R}^{2N}$:

$$\sum_{i=1}^{l} \lambda_i = \lim_{M\to\infty} M^{-1} \ln|\boldsymbol{\zeta}^{(1)}(M) \wedge \ldots \wedge \boldsymbol{\zeta}^{(l)}(M)| \tag{14}$$

where

$$\boldsymbol{\zeta}^{(i)}(M) = \left(\prod_{k=1}^{M} A_\varepsilon(k)\right)\boldsymbol{\zeta}^{(i)}(0) \tag{15}$$

and $A_{ij}=\partial g_i(x(k))/\partial x_j(k)$, $\boldsymbol{\zeta}^{(i)}(0)$ are orthonormal vectors with norms $|\boldsymbol{\zeta}^{(i)}(0)|=1$.

In the framework of the random matrices product the Lyapunov exponents are defined by (14) and (15) with $A_\varepsilon(k)$ given by the rules described in § 2.

The above definition of $\{\lambda_i\}$ is however not useful for practical purposes as it is necessary to perform a Gram-Schmidt orthonormalisation procedure in order to avoid the angle between two vectors $\boldsymbol{\zeta}^{(i)}$ and $\boldsymbol{\zeta}^{(j)}$ becoming too small for numerical computations (see for details Benettin *et al* 1980b).

We have then computed $\{\lambda_i\}$ imposing the constraint (11) on the matrix elements.

Let us remark that $\lambda_{2N+i-1}=-\lambda_i$ as in our case the matrices product is related to a symplectic map; one has besides $\lambda_N=\lambda_{N+1}=0$ because of constraint (11) which implies the conservation law $\Sigma_{i=1}^N P_i=$ constant in system (3). These equalities have been checked in the numerical computation to test the accuracy of the results.

The convergence of λ_i is quite good; the estimated error is 1-3% for $i \leqslant N/2$ and slightly larger (5%) for $i > N/2$ using $M=3\times10^4$.

In figures 3(*a*) and 3(*b*) we show λ_i plotted against i/N for different N values and different probability laws for constructing the matrix elements a_{ij}.

Note that a good approximation of the asymptotic distribution is given for 'large' N by

$$\lambda_i \simeq \lambda_1^*(1 - i/N) \tag{16}$$

where $\lambda_1^* = \lim_{N\to\infty} \lambda_1(N)$. The law (16) seems to hold also without imposing constraint (11) if $\bar{x} = 0$, while the opposite case (i.e. no constraint (11) and $\bar{x} \neq 0$) still leads to an asymptotic distribution but different from (16).

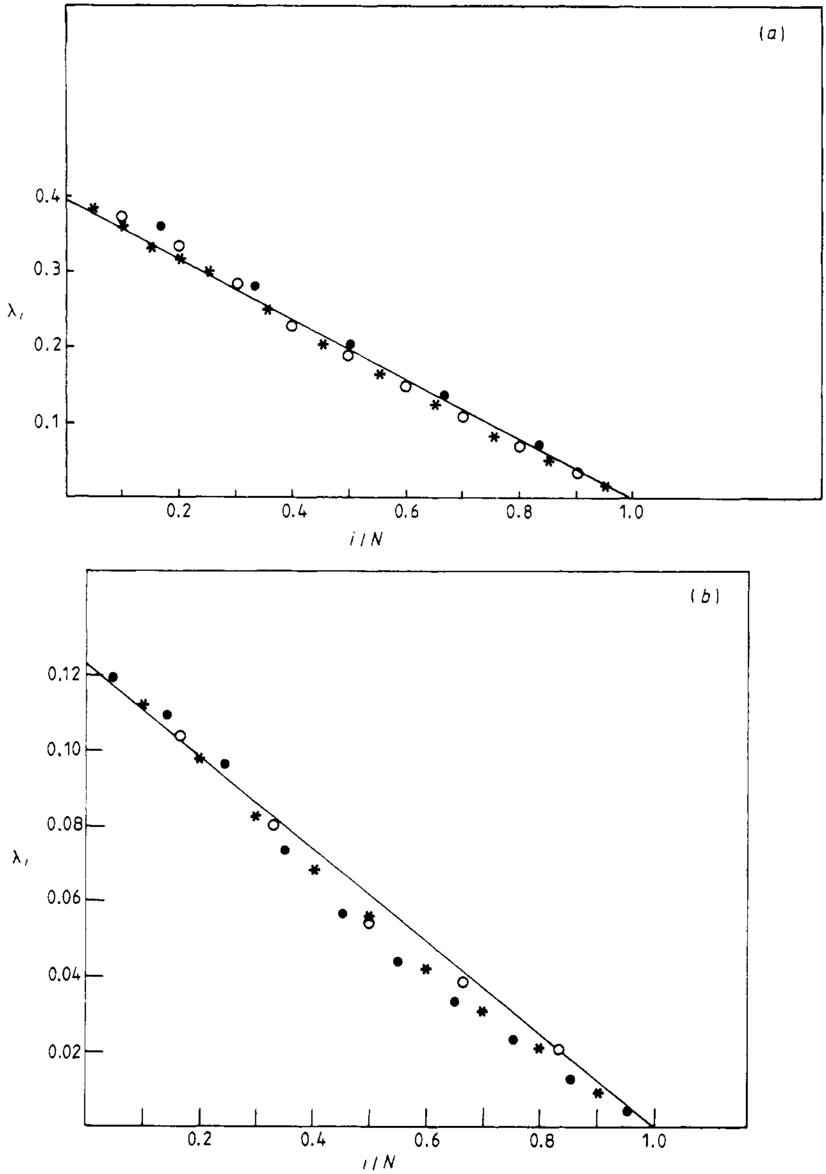

Figure 3. λ_i plotted against i/N at different N imposing constraint (11). (a) $\alpha = 1$, $\bar{x} = 0.5$, $\varepsilon = 1$ and $m = 1$; ●: $N = 6$; ○: $N = 10$; *: $N = 20$. (b) $\alpha = 1$, $\bar{x} = 0$, $\varepsilon = 1$ and $m = 1$; ○: $N = 6$; *: $N = 10$; ●: $N = 20$.

We finally wish to stress that the asymptotic distribution (16) is also obtained by Livi *et al* (1986) for a chain of coupled particles interacting through non-linear forces (Fermi-Pasta-Ulam model). We believe that it is not a coincidence that the asymptotic distribution (16) has been obtained in two quite different Hamiltonian systems. It is almost reasonable enough in the case of 'fully developed chaos' (e.g. for Hamiltonian systems at large energies) to expect a simple asymptotic distribution of Lyapunov characteristic exponents equal (or similar) to (16). For example, Ruelle (1982) has recently found exact bounds for the case of fully developed turbulence. After completion of this work Newman sent us a paper (Newman 1985) where the asymptotic distribution (16) for λ_i is analytically obtained for the infinite product of some particular random matrices. We shall return to this stimulating and intriguing problem with a systematic study of symplectic maps, Hamiltonian and conservative systems.

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References

Arnold V I and Avez A 1968 *Ergodic problems of classical mechanics* (New York: Benjamin)
Benettin G 1984 *Physica* **13D** 211
Benettin G, Galgani L, Giorgilli A and Strelcyn J M 1980a *Meccanica* **15** 9
—— 1980b *Meccanica* **15** 21
Casartelli M, Diana E, Galgani L and Scotti A 1976 *Phys. Rev.* A **13** 1921
Chirikov B V 1982 '*Group theory methods in physics.*' *Talk given at 2nd Int. Seminar Zvenigorod*
Contopoulos G, Galgani L and Giorgilli A 1978 *Phys. Rev.* A **18** 1183
Lichtenberg A J and Lieberman M A 1983 *Regular and Stochastic Motion* (Berlin: Springer)
Livi R, Politi A and Ruffo S 1986 *J. Phys. A: Math. Gen.* **19** 2033
Manneville P 1983 *presented at Burers sur Yvette*
Newman C M 1985 *Commun. Math. Phys.* in press
Pesin Ya B 1976 *Dokl. Akad. Nauk* **226** 774
Pettini M and Vulpiani A 1984 *Phys. Lett.* **106A** 207
Pomeau Y, Pumir A and Pelce P 1984 *J. Stat. Phys.* **37** 39
Rechester A B, Rosenbluth M N and White R B 1979 *Phys. Rev. Lett.* **42** 1247
Ruelle D 1982 *Commun. Math. Phys.* **87** 287